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Research Article

r-Costar Pair of Contravariant Functors

S. Al-Nofayee

Department of Mathematics, Taif University, P.O. Box 439, Hawiah 21974, Saudi Arabia

Correspondence should be addressed to S. Al-Nofayee, alnofayee@hotmail.com

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We generalize r-costar module to r-costar pair of contravariant functors between abelian categories.

1. Introduction

For a ring A, a fixed right A-module M, and $D = \mathrm{End}_A(M)$, let fgd-tl(M_A) denote the class of torsionless right R-modules whose M-dual are finitely generated over D and fg-tl $({}_DM)$ denote the class of finitely generated torsionless left D-modules. M is called costar module if

$$\mathrm{Hom}_A(-, M) : \text{fgd-tl}(M_A) \rightleftarrows \text{fg-tl}({}_DM) : \mathrm{Hom}_D(-, M) \tag{1.1}$$

is a duality. Costar modules were introduced by Colby and Fuller in [1]. M is said to be an r-costar module provided that any exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0, \tag{1.2}$$

such that X and Y are M-reflexive, remains exact after applying the functor $\mathrm{Hom}_A(-, M)$ if and only if Z is M-reflexive. The notion of r-costar module was introduced by Liu and Zhang in [2]. We say that a right A-module X is n-finitely M-copresented if there exists a long exact sequence

$$0 \longrightarrow X \longrightarrow M^{k_0} \longrightarrow M^{k_1} \longrightarrow \cdots \longrightarrow M^{k_{n-1}} \tag{1.3}$$

such that k_i are positive integers for $0 \leq i \leq n-1$. The class of all n-finitely M-copresented modules is denoted by $n\text{-cop}(M)$. We say that a right A-module M is a finitistic n-self-cotilting module provided that any exact sequence

$$0 \longrightarrow X \longrightarrow M^m \longrightarrow Z \longrightarrow 0, \tag{1.4}$$

such that $Z \in n\text{-cop}(M)$ and m is a positive integer, remains exact after applying the functor $\mathrm{Hom}_A(-, M)$ and $n\text{-cop}(Q) = (n+1)\text{-cop}(Q)$. Finitistic n-self-cotilting modules were introduced by Breaz in [3].

In [4] Castaño-Iglesias generalizes the notion of costar module to Grothendieck categories. Pop in [5] generalizes the notion of finitistic n-self-cotilting module to finitistic n-F-cotilting object in abelian categories and he describes a family of dualities between abelian categories. Breaz and Pop in [6] generalize a duality exhibited in [3, Theorem 2.8] to abelian categories.

In this work we continue this kind of study and generalizes the notion of r-costar module to r-costar pair of contravariant functors between abelian categories, by generalizing the work in [2]. We use the same technique of proofs of that paper.

2. Preliminaries

Let $F\colon \mathfrak{C} \to \mathfrak{D}$ and $G\colon \mathfrak{D} \to \mathfrak{C}$ be additive and contravariant left exact functors between two abelian categories $\mathfrak{C}$ and $\mathfrak{D}$. It is said that they are adjoint on the right if there are natural isomorphisms

$$\eta_{X,Y}\colon \mathrm{Hom}_{\mathfrak{C}}(X, G(Y)) \longrightarrow \mathrm{Hom}_{\mathfrak{D}}(Y, F(X)), \tag{2.1}$$

for $X \in \mathfrak{C}$ and $Y \in \mathfrak{D}$. Then they induce two natural transformations $\delta : 1_{\mathfrak{C}} \to GF$ and $\delta' : 1_{\mathfrak{D}} \to FG$ defined by $\delta_X = \eta^{-1}_{X,F(X)}(1_{F(X)})$ and $\delta'_Y = \eta^{-1}_{G(Y),Y}(1_{G(Y)})$. Moreover the following identities are satisfied for each $X \in \mathfrak{C}$ and $Y \in \mathfrak{D}$:

$$F(\delta_X) \circ \delta'_{F(X)} = 1_{F(X)}, \qquad G(\delta'_Y) \circ \delta'_{G(Y)} = 1_{G(Y)}. \tag{2.2}$$

The pair (F, G) is called a duality if there are functorial isomorphisms $GF \simeq 1_{\mathfrak{C}}$ and $FG \simeq 1_{\mathfrak{D}}$. An object X of $\mathfrak{C}$ ($Y \in \mathfrak{D}$) is called *F-reflexive* (resp., *G-reflexive*) in case δ_X (resp., δ'_Y) is an isomorphism. By $\mathrm{Ref}(F)$ we will denote the full subcategory of all F-reflexive objects. As well by $\mathrm{Ref}(G)$ we will denote the full subcategory of all G-reflexive objects. It is clear that the functors F and G induce a duality between the categories $\mathrm{Ref}(F)$ and $\mathrm{Ref}(G)$.

We say that the pair (F, G) of left exact contravariant functors is r-costar provided that any exact sequence

$$0 \longrightarrow Q \longrightarrow U \longrightarrow V \longrightarrow 0, \tag{2.3}$$

with $Q, U \in \mathrm{Ref}(F)$ remains exact after applying the functor F if and only if $V \in \mathrm{Ref}(F)$.

An object U is called V-finitely generated if there is an epimorphism $V^n \to X \to 0$, for some positive integer n. We denote by $\mathrm{gen}(V)$ the subcategory of all V-finitely generated objects. $\mathrm{add}(V)$ denotes the class of all summands of finite direct sums of copies of V. We will denote by $\mathrm{proj}(\mathfrak{D})$ the full subcategory of all projective objects in $\mathfrak{D}$.

From now on we suppose that $\mathfrak{D}$ has enough projectives that is, for every object $X \in \mathfrak{D}$ there is a projective object $P \in \mathfrak{D}$ and an epimorphism $P \to X \to 0$. It is clear that we can construct a projective resolution for any object X. Suppose we have a projective resolution of X

$$P\colon \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0. \tag{2.4}$$

This gives rise to the sequence

$$0 \longrightarrow G(X) \longrightarrow G(P_0) \longrightarrow G(P_1) \longrightarrow G(P_2) \longrightarrow \cdots, \tag{2.5}$$

and the cochain complex $G(P)$, which we can compute its cohomology at the nth spot (the kernel of the map from $G(P_n)$ modulo the image of the map to $G(P_n)$) and denote it by $H^n(G(P))$. We define $R^nG(X) = H^n(G(P))$ as the nth right derived functor of G. For the functor G we define ${}^{\perp}T_G^{i\geqslant n} = \{X \in \mathfrak{D} : R^iG(X) = 0 \text{ for every } i \geqslant n\}$.

Let

$$0 \longrightarrow Q \stackrel{f}{\longrightarrow} U \longrightarrow V \longrightarrow 0 \tag{2.6}$$

be an exact sequence in $\mathfrak{C}$. Applying the functor F we get the exact sequence

$$0 \longrightarrow F(V) \longrightarrow F(U) \stackrel{p}{\longrightarrow} X \longrightarrow 0, \tag{2.7}$$

where $X = \mathrm{Im}(F(f))$. Let $F(f) = j \circ p$ be the canonical decomposition of $F(f)$, where $j\colon X \to F(Q)$ is the inclusion map. Applying the functor G to the sequence (2.7), we have the following exact sequence

$$0 \longrightarrow GX \stackrel{G(p)}{\longrightarrow} GF(U) \longrightarrow GF(V). \tag{2.8}$$

Now if we put $\alpha = G(j) \circ \delta_Q$, then

$$G(p) \circ \alpha = G(p) \circ G(j) \circ \delta_Q = G(j \circ p) \circ \delta_Q = GF(f) \circ \delta_Q = \delta_U \circ f. \tag{2.9}$$

So we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q & \stackrel{f}{\longrightarrow} & U & \longrightarrow & V & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta_U} & & \downarrow{\scriptstyle \delta_V} & & \\
0 & \longrightarrow & GX & \stackrel{G(p)}{\longrightarrow} & GF(U) & \longrightarrow & GF(V) & &
\end{array} \tag{2.10}$$

3. r-Costar Pair of Contravariant Functors

We will fix all the notations and terminologies used in previous section.

Proposition 3.1. *Let (F, G) be a pair of left exact contravariant functors which are adjoint on the right. Assume that* $\mathrm{Ref}(G) \subseteq {}^{\perp}T_G^{i\geqslant 1}$ *and* ${}^{\perp}T_G^{i\geqslant 0} = 0$*. Then (F, G) is an r-costar.*

Proof. Let

$$0 \longrightarrow Q \xrightarrow{f} U \longrightarrow V \longrightarrow 0 \tag{3.1}$$

be an exact sequence with $Q, U \in \mathrm{Ref}(F)$. Assume that we have the exact sequence

$$0 \longrightarrow F(V) \longrightarrow F(U) \longrightarrow F(Q) \longrightarrow 0, \tag{3.2}$$

after applying the functor F. Applying the functor G to the last sequence, we get an exact sequence

$$0 \longrightarrow GF(Q) \longrightarrow GF(U) \longrightarrow GF(V) \longrightarrow R^1G(F(Q)) = 0, \tag{3.3}$$

since $F(Q) \in \mathrm{Ref}(G) \subseteq {}^{\perp}T_G^{i\geqslant 1}$. Hence we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q & \longrightarrow & U & \longrightarrow & V & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \delta_Q} & & \downarrow{\scriptstyle \delta_U} & & \downarrow{\scriptstyle \delta_V} & & \downarrow \\ 0 & \longrightarrow & GF(Q) & \longrightarrow & GF(U) & \longrightarrow & GF(V) & \longrightarrow & 0 \end{array} \tag{3.4}$$

Since $Q, U \in \mathrm{Ref}(F)$, δ_Q and δ_U are isomorphisms. Now it is clear that δ_V is an isomorphism which means that $V \in \mathrm{Ref}(F)$.

Conversely, suppose that $V \in \mathrm{Ref}(F)$. Applying the functor F to the sequence (3.1), we get an exact sequence

$$0 \longrightarrow F(V) \longrightarrow F(U) \longrightarrow X \longrightarrow 0, \tag{3.5}$$

where $X = \mathrm{Im}(F(f))$. Hence we can get the exact sequence

$$0 \longrightarrow X \xrightarrow{j} F(Q) \longrightarrow Y \longrightarrow 0, \tag{3.6}$$

for some $Y \in \mathfrak{D}$, and j is the inclusion map. Applying the functor G to the sequence (3.5), we have the following exact commutative diagram (see diagram (2.10))

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Q & \longrightarrow & U & \longrightarrow & V & \longrightarrow & 0 & & \\ \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \delta_U} & & \downarrow{\scriptstyle \delta_V} & & \downarrow & & \\ 0 & \longrightarrow & GX & \longrightarrow & GF(U) & \longrightarrow & GF(V) & \longrightarrow & R^1G(X) & \longrightarrow & R^1G(F(U)) = 0 \end{array} \tag{3.7}$$

where $\alpha = G(j) \circ \delta_Q$. Note that δ_U and δ_V are isomorphisms, since Q, $U \in \operatorname{Ref}(F)$. It is clear from the diagram that $R^1G(X) = 0$. Now $R^iG(X) = 0$ for all $i \geqslant 2$, by dimension shifting, since $F(U), F(V) \in \operatorname{Ref}(G) \subseteq {}^{\perp}T_G^{i\geqslant 1}$. Hence $X \in {}^{\perp}T_G^{i\geqslant 1}$. Now applying the functor G to sequence (3.6), we get the long exact sequence

$$\begin{aligned} 0 &\longrightarrow G(Y) \longrightarrow GF(Q) \xrightarrow{G(j)} G(X) \longrightarrow R^1G(Y) \\ &\longrightarrow R^1G(F(Q)) \longrightarrow R^1G(X) \longrightarrow R^2G(Y) \longrightarrow R^2G(\mathrm{F}(Q)) \longrightarrow \cdots . \end{aligned} \tag{3.8}$$

Above we conclude that $X \in {}^{\perp}T_G^{i\geqslant 1}$ and by assumptions $F(Q) \in \operatorname{Ref}(G) \subseteq {}^{\perp}T_G^{i\geqslant 1}$, thus $R^1G(X) = 0$ and $R^2G(F(Q)) = 0$. Hence by dimension shifting $Y \in {}^{\perp}T_G^{i\geqslant 2}$. Now consider the following part from sequence (3.8)

$$0 \longrightarrow G(Y) \longrightarrow GF(Q) \xrightarrow{G(j)} G(X) \longrightarrow R^1G(Y). \tag{3.9}$$

Note that $\alpha = G(j) \circ \delta_Q$ in diagram (3.7) is an isomorphism, since δ_U and δ_V are isomorphisms. Hence $G(j)$ is an isomorphism, since δ_Q is an isomorphism, so from sequence (3.9), $R^1G(Y) = 0 = G(Y)$. We conclude that $Y \in {}^{\perp}T_G^{i\geqslant 0}$. Since ${}^{\perp}T_G^{i\geqslant 0} = 0$ by assumptions, $Y = 0$ and hence from sequence (3.6) $X \cong F(Q)$ canonically. Therefore the functor F preserves the exactness of the exact sequence

$$0 \longrightarrow Q \longrightarrow U \longrightarrow V \longrightarrow 0 \tag{3.10}$$

in $\operatorname{Ref}(F)$. We conclude that the pair (F, G) is an r-costar. □

Corollary 3.2. *Let (F, G) be a pair of left exact contravariant functors which are adjoint on the right. Assume that* $\operatorname{Ref}(G) = {}^{\perp}T_G^{i\geqslant 1}$. *Then (F, G) is an r-costar.*

Proof. Let $X \in {}^{\perp}T_G^{i\geqslant 0}$, then $X \in {}^{\perp}T_G^{i\geqslant 1} = \operatorname{Ref}(G)$. Hence $X \cong FG(X) = 0$. □

Proposition 3.3. *Let V be a G-reflexive generator in $\mathfrak{D}$ and $U = G(V)$. Let (F, G) be an r-costar pair. If* $\operatorname{Ref}(G) \subseteq g(V)$, *then for any* $X \in \operatorname{Ref}(F)$, *there is an infinite exact sequence*

$$0 \longrightarrow X \longrightarrow U_1 \longrightarrow \cdots \longrightarrow U_n \longrightarrow \cdots , \tag{3.11}$$

which remains exact after applying the functor F, where $U_i \in \operatorname{add}(U)$ for each i.

Proof. Let $X \in \operatorname{Ref}(F)$. Then $F(X) \in \operatorname{Ref}(G)$, so by assumption there is an exact sequence

$$V^n \longrightarrow F(X) \longrightarrow 0. \tag{3.12}$$

Applying the functor G we have an exact sequence

$$0 \longrightarrow X \longrightarrow U^n \longrightarrow Y \longrightarrow 0, \tag{3.13}$$

for some $Y \in \mathfrak{C}$. Since (F, G) is an r-costar pair, the last sequence is exact after applying the functor F, that is we have an exact sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^n) \longrightarrow F(X) \longrightarrow 0. \tag{3.14}$$

Applying the functor G again we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & U^n & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \delta_{U^n}} & & \downarrow{\scriptstyle \delta_Y} & & \\ 0 & \longrightarrow & GF(X) & \longrightarrow & GF(U^n) & \longrightarrow & GF(Y) & & \end{array} \tag{3.15}$$

Since $X, U^n \in \operatorname{Ref}(F)$, $Y \in \operatorname{Ref}(F)$. By repeating the process to Y, and so on, we finally obtain the desired exact sequence. □

Proposition 3.4. *Let V be a G-reflexive and a projective generator in $\mathfrak{D}$ and $U = G(V)$. Let (F, G) be an r-costar pair and suppose that* $\operatorname{Ref}(G) \subseteq \operatorname{gen}(V)$. *Then* $\operatorname{Ref}(G) \subseteq {}^{\perp}T_G^{i \geqslant 1}$.

Proof. Let $X \in \operatorname{Ref}(G)$, then $G(X) \in \operatorname{Ref}(F)$ and hence by Proposition 3.3, there is an infinite exact sequence

$$0 \longrightarrow G(X) \longrightarrow U_1 \longrightarrow \cdots \longrightarrow U_n \longrightarrow \cdots, \tag{3.16}$$

which remains exact after applying the functor F, where $U_i \in \operatorname{add}(U)$ for each i. So we have an exact sequence

$$\cdots \longrightarrow F(U_n) \longrightarrow \cdots \longrightarrow F(U_1) \longrightarrow FG(X) \longrightarrow 0 \tag{3.17}$$

Again the last sequence remains exact after applying the functor G, since we get a sequence isomorphic to sequence (3.16), because $G(X)$, U_i, for each i, are F-reflexive. We obtain that $\operatorname{Ref}(G) \subseteq {}^{\perp}T_G^{i \geqslant 1}$ by dimension shifting. □

Suppose we have the following exact sequence in $\mathfrak{D}$

$$0 \longrightarrow X \longrightarrow P_2 \longrightarrow P_1 \longrightarrow Y \longrightarrow 0, \tag{3.18}$$

where P_2, P_1 are projective objects in $\mathfrak{D}$ and $Y \in {}^{\perp}T_G^{i \geqslant 1}$. Applying the functor G we get the following exact sequence

$$0 \longrightarrow G(Y) \longrightarrow G(P_1) \longrightarrow G(P_2) \longrightarrow G(X) \longrightarrow R^1G(Y) = 0. \tag{3.19}$$

Applying the functor F we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \longrightarrow & P_2 & \longrightarrow & P_1 \\ \downarrow & & \downarrow \delta_X & & \downarrow \delta_{P_2} & & \downarrow \delta_{P_1} \\ 0 & \longrightarrow & FG(X) & \longrightarrow & FG(P_2) & \longrightarrow & FG(P_1) \end{array} \tag{3.20}$$

If $\operatorname{proj}(\mathfrak{D}) \subseteq \operatorname{Ref}(G)$, then it is clear that $X \in \operatorname{Ref}(G)$.

Proposition 3.5. *Let V be a G-reflexive generator in $\mathfrak{D}$. Let (F, G) be an r-costar pair and suppose that* $\operatorname{proj}(\mathfrak{D}) \subseteq \operatorname{Ref}(G) \subseteq \operatorname{gen}(V)$. Then ${}^{\perp}T_G^{i\geqslant 0} = 0$.

Proof. For any $X \in {}^{\perp}T_G^{i\geqslant 0}$, we can build the following exact sequence in $\mathfrak{D}$

$$0 \longrightarrow Y \longrightarrow P_2 \longrightarrow P_1 \longrightarrow X \longrightarrow 0, \tag{3.21}$$

where P_2, P_1 are projective objects and Y an object in $\mathfrak{D}$. By the argument before the proposition it is clear that $Y \in \operatorname{Ref}(G)$ and hence $G(Y) \in \operatorname{Ref}(F)$. Applying the functor G we get the following exact sequence

$$0 \longrightarrow G(X) = 0 \longrightarrow G(P_1) \longrightarrow G(P_2) \longrightarrow G(Y) \longrightarrow R^1G(X) = 0. \tag{3.22}$$

Applying the functor F we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & Y & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & X & \longrightarrow & 0 \\ \downarrow & & \downarrow \delta_Y & & \downarrow \delta_{P_2} & & \downarrow \delta_{P_1} & & \downarrow & & \\ 0 & \longrightarrow & FG(Y) & \longrightarrow & FG(P_2) & \longrightarrow & FG(P_1) & \longrightarrow & 0 & & \end{array} \tag{3.23}$$

Thus it is clear that $X \cong 0$. □

Now we are able to give the following characterization of r-costar pair.

Theorem 3.6. *Let (F, G) be a pair of left exact contravariant functor which are adjoint on the right. Suppose that V be a G-reflexive projective generator in $\mathfrak{D}$ and* $\operatorname{proj}(\mathfrak{D}) \subseteq \operatorname{Ref}(G) \subseteq \operatorname{gen}(V)$. *Then (F, G) is an r-costar if and only if* $\operatorname{Ref}(G) \subseteq {}^{\perp}T_G^{i\geqslant 1}$ *and* ${}^{\perp}T_G^{i\geqslant 0} = 0$.

Proof. By Propositions 3.4, 3.1, and 3.5. □

Corollary 3.7. *Let (F, G) be a pair of left exact contravariant functor which are adjoint on the right. Suppose that V be a G-reflexive projective generator in $\mathfrak{D}$ and $U = G(V)$. If* $\operatorname{proj}(\mathfrak{D}) \subseteq \operatorname{Ref}(G) \subseteq \operatorname{gen}(V)$, *then the following are equivalent.*

(1) *(F, G) is an r-costar.*

(2) *For any exact sequence*

$$0 \longrightarrow X \longrightarrow U^n \longrightarrow Y \longrightarrow 0, \tag{3.24}$$

with $X \in \mathrm{Ref}(F)$, *then* $Y \in \mathrm{Ref}(F)$ *if and only if the exact sequence remains exact after applying the functor* F.

Proof. (1)⇒(2) follows from the definition of r-costar pair.

(2)⇒(1) the proof goes the same as the proofs of Propositions 3.3, 3.4, 3.5, and Theorem 3.6. □

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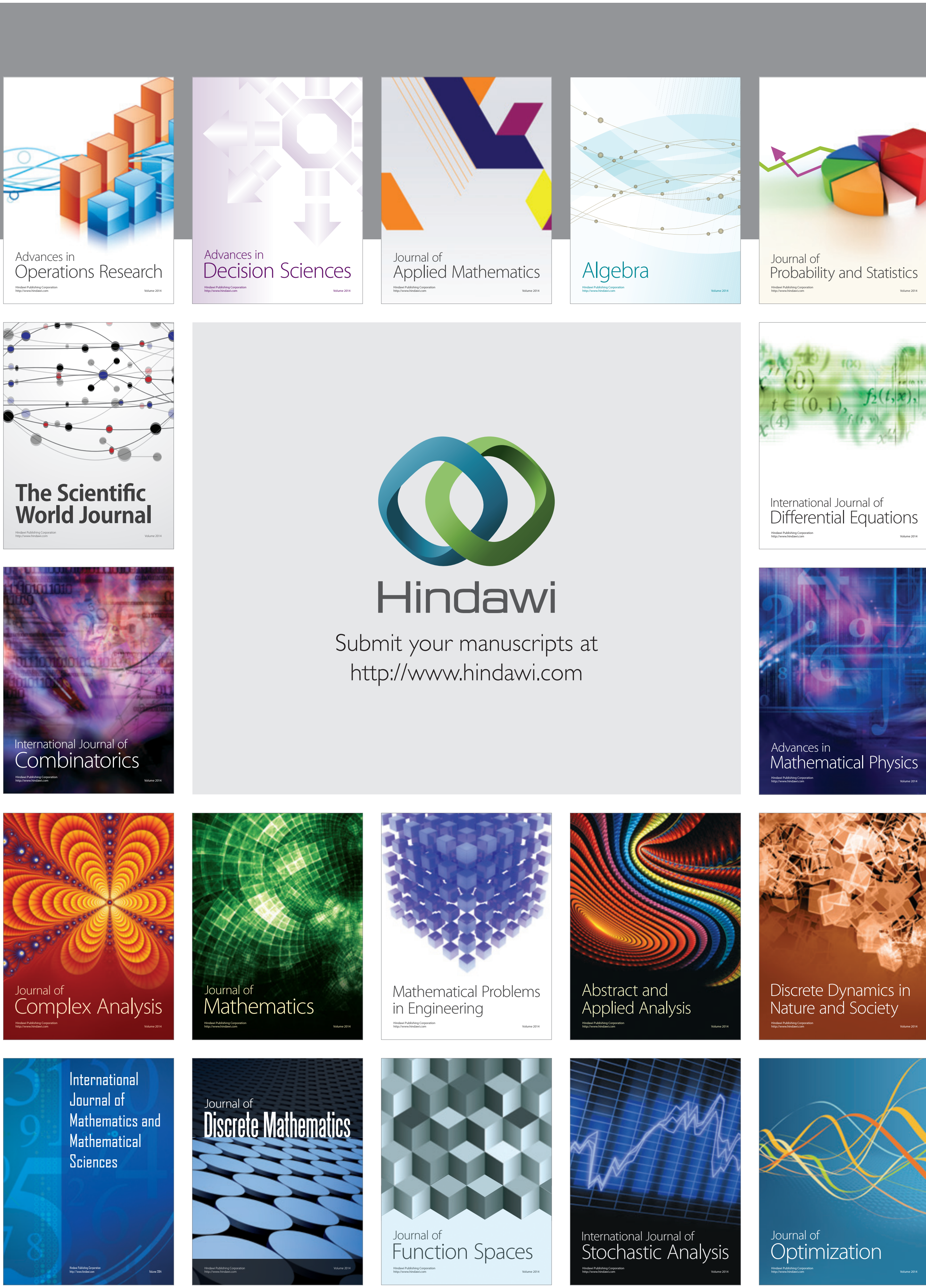
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